The Role of Shear Forces in Arterial Branching

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ABSTRACT A new optimality principle for the branching angles of blood vessels in the cardiovascular system is proposed: the principle of minimum drag. The results are examined in the light of general observations and compared with those obtained from the principles of minimum work and minimum volume. It is shown that in some aspects the new principle is equally consistent with observations, and, in other aspects, it is perhaps more plausible than the other two principles.

INTRODUCTION

The branching geometry of blood vessels in the cardiovascular system has been the subject of much study and speculation, the general aim being to establish the principles governing the angles at a point where a blood vessel gives rise to one or more branches. An understanding of this aspect of the cardiovascular system is important not only for its intrinsic or aesthetic value, but also for the bearing it may have on our understanding of the design and function of the cardiovascular system in general and some of its anomalies in particular. Examples of the latter are aneurysms (Roach et al., 1972), intimal cushions (Hassler, 1962), atherosclerotic plaques (Fry, 1968), the Fahraeus-Lindqvist effect (Zamir and Roach, 1972, 1973 a) and other blood flow problems (Zamir and Roach, 1973 b). Early work on the branching geometry of blood vessels was characterized mainly by qualitative studies. These studies and the related physiological arguments were reviewed eloquently by Sir D’Arcy Thompson in 1942.

An attempt to quantify these arguments was made by Murray (1926 b). In an earlier paper, Murray (1926 a) used the principle of minimum work to predict the diameters of blood vessels and then, using the results, he applied the same principle to predict the branching angles in his second paper. The basic argument in both instances was that the diameters of blood vessels and the angles at which their branches arise are such that the work required to circulate and maintain the blood is a minimum. This and similar arguments have since been employed by a number of authors in different contexts. For example, it was used by Murray (1927) to predict the branching angles in trees, by Turner (1968) to explain the geometry of the human cerebellar vermis, and by Leopold (1971) to explain the geometry of the drainage streams of rivers. Nevertheless, Murray’s original work on the branching angles of blood vessels has not altogether been received with either the enthusiasm or the criticism which it deserves. Some of the reasons for this it is hoped will become apparent at various points in the present paper.

More recently, Kamiya and Togawa (1972) proposed the alternative view that the diameters of blood vessels and their branching angles are such that the total volume of blood in the cardiovascular system is a minimum. It was also argued that in addition to its obvious advantages, the property of minimum blood volume would also ensure that “the transmission times of informations through the circulation delivered by hormones, oxygen, carbon dioxide and other blood solutes are shortened by as much as the blood volume decreases, because the circulation time of blood is given by volume/flow.”

A major difficulty with both the minimum work and the minimum volume principles is that they offer no feasible mechanism whereby such principles can be implemented. In the cardiovascular system of man, for example, the number of branch points is at least one billion. If the branching angles at these points were indeed governed by the principle of minimum work or that of minimum volume, then it must follow that either (a) the vessel tissue at each one of a billion branch points is capable of “sensing” either work or volume and then adjusting its geometry accordingly so as to minimize either one or the other, or (b) the branching geometry at each one of these billion points is predetermined genetically. The first of these implications is clearly not feasible and almost inconceivable. The second implication is very unlikely in view of the high degree of variation in geometrical details of the cardiovascular system from one individual to another. In any case, there is certainly no experimental evidence in support of either one or the other of these implications. In fact, the normal embryonic development of the cardiovascular system (Zamir and Roach, 1975), and abnormal modifications of the fully developed system by surgery or by the presence of cancerous tissue (Warren, 1968), suggest rather strongly that local geometrical details of the system are determined by local factors and on a local ad hoc basis.

It thus appears likely that the principle governing the branching angles of blood vessels is associated with a key factor which must be mechanical in nature, and more direct and local in its influence, in order to facilitate the mechanism whereby the principle can be implemented at a large number of branch points. The most likely candidate for this role is the shear force between the blood and vessel tissue. As will be shown later, this force is related to flow, vessel diameters, vessel lengths, and, hence, branching angles. The purpose of this paper is to propose and pursue the hypothesis that the branching angles of blood vessels in the cardiovascular system are such that the total shear force on the parent vessel and its branches is a minimum. For brevity, and for reasons to be explained in the next section, this shall be referred to as the principle of minimum drag.

THE CONCEPT OF MINIMUM DRAG

In a fully developed Poiseuille flow, the shear stress $\tau$ exerted by the fluid on the inner surface of the tube is given by

$$\tau = 4\eta f/\pi r^3,$$

where $\eta$ is the viscosity of the fluid, $f$ is the volumetric flow rate, and $r$ is the radius of the tube. The total shear force $T$ on a segment of length $l$ of the tube is then given by:
and if \( t \) denotes the shear force per unit length of the tube, we can write this as
\[
T = tl
\]
where
\[
t = \frac{8\eta l}{r^2}.
\]

The direction of \( T \) is tangential to the inner surface of the tube and parallel to the direction of the flow. Thus as a result of this shear, the tube experiences a force \( T \) which tends to "drag" it along with the flow. The name "drag force" for \( T \) is therefore appropriate in this context. If the tube is free from any external constraints, it will move under the action of this drag force. In the case of a blood vessel however, tethering forces on the outer surface of the vessel prevent this movement to a large extent, though not completely. If the vessel is tethered completely and uniformly at every point of its outer surface, only the inner layers of the vessel tissue will undergo the very small movement allowed by the elasticity of that tissue. The net result in this case is a shear stress within the vessel wall, created by the drag force on the inner surface and the opposing tethering force on the outer surface. However, blood vessels are in general not uniformly tethered, and in this case, the net result is a shear stress within the vessel wall only at points where the vessel is tethered, plus a tension force within the vessel wall between tethering points. This tension force is equal to the corresponding drag force between the same tethering points. Thus if a blood vessel is tethered at two sections \( S_0 \) and \( S_1 \) distant \( l \) apart, and if the flow direction is from \( S_0 \) to \( S_1 \), then the drag force \( T \) on the intervening segment of the vessel will cause a longitudinal tension in the vessel wall which will be experienced by \( S_0 \) as a force \( T \) tending to drag it along the flow.

In the case of a bifurcation, such as that shown in Fig. 1, it is reasonable to assume that at least during the formation and development of the system, the vessels will not be tethered at the junction point but at some distance away from it in order to give that junction the freedom to reach its optimum geometry. Thus if the parent and two daughter vessels are tethered at sections \( S_0, S_1, S_2 \), distant \( l_0, l_1, l_2 \), respectively, from the junction point, and if the flow direction is from parent to daughter vessels, then the total drag force \( T \) on the three intervening vessel segments will be experienced by \( S_0 \) as a force \( T \) tending to drag it along the flow in the parent vessel. This force is a direct and tangible effect on the parent vessel, and it is therefore reasonable to at least examine the hypothesis that the bifurcation will reach its final and optimum geometry when the total drag force \( T \) at \( S_0 \) is a minimum. Thus unlike the minimum work or minimum volume principles, the principle of minimum drag has a direct and local mechanism whereby it can be implemented.

**Analysis of Minimum Drag**

Let \( r_0, r_1, r_2 \) be the radii of and \( f_0, f_1, f_2 \) the flows in the three vessels forming the bifurcation shown in Fig. 1. The corresponding drag forces \( T_0, T_1, T_2 \) are then given by, using Eq. 3,
Figure 1. An arterial bifurcation model in which the vessels are tethered at sections $S_0, S_1, S_2$ distant $l_0, l_1, l_2$, respectively, from the junction point. The branching angles of the two daughter vessels are $\theta_1$ and $\theta_2$.

$$T_0 = t_0 d_0, \quad T_1 = t_1 d_1, \quad T_2 = t_2 d_2,$$

where

$$t_0 = \frac{8\eta f_0 r_0^2}{r_0^2}, \quad t_1 = \frac{8\eta f_1 r_1^2}{r_1^2}, \quad t_2 = \frac{8\eta f_2 r_2^2}{r_2^2}.$$ (4)

As before, $l_0, l_1, l_2$ are the distances, respectively, of the tethering sections $S_0, S_1, S_2$ from the junction point.

The total drag force $T$ acting at $S_0$ is then simply the sum of the three individual drag forces, whatever the angles $\theta_1$ and $\theta_2$ are, i.e.

$$T = T_0 + T_1 + T_2$$

$$= t_0 d_0 + t_1 d_1 + t_2 d_2.$$ (5)

The reason for this somewhat surprising result is that, as explained in detail before, the drag forces $T_1$ and $T_2$ are transmitted to $S_0$ by means of longitudinal tension in the vessel walls. In the absence of any tethering in the junction region, therefore, these forces are transmitted in full rather than in terms of their components along $l_0$. The situation is strictly analogous to that of the transmission of a force by means of a string on a smooth frictionless pulley. A mechanical analogy of this type for the whole bifurcation is shown in Fig. 2.

Let $A_0, A_1, A_2$ be the center points of the tethering sections $S_0, S_1, S_2$, respectively, and consider a two-dimensional rectangular Cartesian coordinate system in which $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are the position coordinates of those three points, and $(x, y)$ are the position coordinates of the junction point. From simple geometry, substituting for $l_0, l_1, l_2$ in Eq. 5 we get

$$T = t_0 \{ (x - x_0)^2 + (y - y_0)^2 \}^{\frac{1}{4}} + t_1 \{ (x - x_1)^2 + (y - y_1)^2 \}^{\frac{1}{4}} + t_2 \{ (x - x_2)^2 + (y - y_2)^2 \}^{\frac{1}{4}}.$$ (6)

The total drag force $T$ thus depends on the position of the junction point $(x, y)$ and as the position of this point changes, with $A_0, A_1, A_2$ remaining fixed, the
branching geometry changes accordingly and with it the drag force $T$. We seek a particular position of the junction point $(x, y)$, i.e. a particular branching geometry, for which $T$ is a minimum. This is easily obtained by setting the partial derivatives of $T$ with respect to $x$ and $y$ equal to zero. Thus, using Eq. 6 we get

$$
\frac{\partial T}{\partial x} = \frac{t_0 (x - x_0)}{(x - x_0)^2 + (y - y_0)^2} + t_1 \frac{(x - x_1)}{(x - x_1)^2 + (y - y_1)^2} + t_2 \frac{(x - x_2)}{(x - x_2)^2 + (y - y_2)^2},
$$

(7)

and, from simple geometry again, these can be simplified into

$$
\frac{\partial T}{\partial y} = \frac{t_0 (y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + t_1 \frac{(y - y_1)}{(x - x_1)^2 + (y - y_1)^2} + t_2 \frac{(y - y_2)}{(x - x_2)^2 + (y - y_2)^2},
$$

(8)

where the angles $\theta_0$, $\theta_1$, $\theta_2$ are as indicated in Fig. 1. Equating these two derivatives to zero, we obtain the following conditions for minimum drag force $T$:

$$
\begin{align*}
\frac{t_0 \cos \theta_0 - t_1 \cos (\theta_1 + \theta_0) - t_2 \cos (\theta_2 - \theta_0)}{\partial x} &= 0, \\
\frac{t_0 \sin \theta_0 - t_1 \sin (\theta_1 + \theta_0) + t_2 \sin (\theta_2 - \theta_0)}{\partial y} &= 0.
\end{align*}
$$

(9)
In the corresponding analysis to obtain the conditions for minimum work, Murray (1926 b) used differentials in place of derivatives and made an appeal to the principle of virtual work. Also, the word "work" was used throughout to mean "power." Thus, while his final results are correct, Murray's analysis and notation will appear to be somewhat different and may require a certain amount of clarification in places. From Eqs. 9, finally, by squaring \( t_1\cos(\theta_1 + \theta_0) \) and \( t_1\sin(\theta_1 + \theta_0) \) and adding, we obtain
\[
t_1^2 = t_0^2 + t_2^2 - 2t_0t_2\cos\theta_2,
\]
and by squaring \( t_2\cos(\theta_2 - \theta_0) \) and \( t_2\sin(\theta_2 - \theta_0) \) and adding, we obtain
\[
t_2^2 = t_0^2 + t_1^2 - 2t_0t_1\cos\theta_1.
\]
Thus the branching geometry is optimum when the branching angles are given by
\[
\cos\theta_1 = (t_0^2 + t_1^2 - t_2^2)/2t_0t_1,
\]
\[
\cos\theta_2 = (t_0^2 + t_2^2 - t_1^2)/2t_0t_2.
\]
As expected, the final results do not depend on \( \theta_0 \) since that angle, between the parent vessel and the \( x \) axis, depends on the choice of coordinate system only and it is therefore strictly arbitrary.

**RESULTS**

The optimum branching angles for minimum total drag \( T \) can be expressed in terms of vessel radii and flows, using Eqs. 4 and 10, to give
\[
2\cos\theta_1 = (F_1/R_1^2) + (F_2/R_2^2) - (F_1F_2/R_1R_2^2),
\]
\[
2\cos\theta_2 = (F_1/R_1^2) + (F_2/R_2^2) - (F_1F_2/R_2R_1^2).
\]
It is more convenient to express these in terms of the following nondimensional ratios of radii and flows
\[
R_1 = r_2/r_0,
\]
\[
R_2 = r_1/r_0,
\]
\[
F_1 = f_1/f_0,
\]
\[
F_2 = f_2/f_0.
\]
Substituting these into Eq. 11 we get
\[
2\cos\theta_1 = (R_1^2F_1) + (F_1R_2^2) - (F_1R_2F_2R_1^2),
\]
\[
2\cos\theta_2 = (R_2^2F_2) + (F_2R_1^2) - (F_1R_2F_2R_1^2).
\]
These can be combined to yield an expression for the total bifurcation angle
\[
\cos(\theta_1 + \theta_2) = (R_1^2R_2^2 - F_1^2R_2^2 - F_1^2R_1^2)/2(F_1F_2R_1^2R_2^2).
\]
From Eqs. 13, we note for later reference that as
\[
F_1, R_1 \to 1
\]
and
\[
F_2, R_2 \to 0;
\]
\[
\theta_1 \to 0
\]
Similarly, as
\[ F_1, R_1, (F_1/R_1) \to 0\]
and
\[ F_2, R_2 \to 1; \]
\[ \theta_1 \to 90^\circ \]
and
\[ \theta_2 \to 0. \]

In the case of a symmetrical bifurcation where \( F_1 = F_2 = \frac{1}{2} \) writing
\[ R_1 = R_2 = R \]
and
\[ \theta_1 = \theta_2 = \theta, \]
Eq. 14 reduces to
\[ \cos 2\theta = 2R^4 - 1. \] (18)

Introducing the commonly used “area ratio” \( \beta \), which in general is defined as
\[ \beta = (\frac{\pi r_1^2 + \pi r_2^2}{\pi r_0^2}) \]
\[ = R_1^2 + R_2^2, \] (19)

and in the case of a symmetrical bifurcation is given by
\[ \beta = 2R^2, \] (20)

Eq. 18 can be put in the form
\[ \cos 2\theta = (\beta^{1/2}) - 1. \] (21)

This result is shown graphically in Fig. 3.

**Figure 3.** The relation between total bifurcation angle (2θ) and area ration (β) for a symmetrical bifurcation as given by Eq. 21.
DISCUSSION AND CONCLUSIONS

From the results in Eqs. 15 and 16 it may be concluded in qualitative terms that when a parent blood vessel undergoes a bifurcation, the larger branch makes a smaller angle with the direction of the parent vessel than does the smaller branch. In particular, if one branch is very much larger than the other, then the former will appear like a continuation of the parent vessel while the latter will seem to rise at a right angle.

While these conclusions were here derived from the principle of minimum drag, it is highly significant that the same conclusions can be and have been derived from the principle of minimum work and from that of minimum volume. It may be strongly suggested, therefore, that such qualitative conclusions cannot be used as they have been in the past to establish the validity of any particular one of these optimality principles. Evidently, the suggested branching pattern is in good qualitative agreement with the pattern observed in the cardiovascular system. On the basis of this evidence, however, it would be safer and certainly more accurate to conclude only that an optimality principle of some kind is at work in the branching pattern of the cardiovascular system.

From the results in Eq. 21 and Fig. 3 it may be concluded that in the case of a symmetrical bifurcation, and perhaps in general, the total branching angle decreases as the area ratio increases. This result has not been obtained previously and it is not inconsistent with the general pattern observed in the cardiovascular system, in the absence of any accurate measurements. In contrast, the analyses of both Murray (1926 b) and Kamiya and Togawa (1972) lead to the result that the area ratio in all symmetrical bifurcations is the same and equal to approximately 1.26. This result is certainly not supported by whatever evidence is available and it is in fact in strong disagreement with the measurements of Caro et al. (1971). Also, fluid dynamic considerations (Lighthill, 1972) suggest that there are good reasons for which the area ratio should vary throughout the cardiovascular system. The present results provide the relation between area ratio and total bifurcation angle, which it is hoped will be tested in the light of future data.

The results of the present study, Eqs. 15, 16, and 21, can also be supplemented by an assumption regarding the optimum relation between flow (f) and vessel radius (r) in the cardiovascular system in general. Such an assumption was in fact made by Murray (1926 b) and one was implied by Kamiya and Togawa (1972) and, in both cases, a specific relation between f and r was incorporated with the optimality principle for branching angles. It is this additional assumption which led those authors to the result that the area ratio is constant and equal to 1.26 in a symmetrical bifurcation. The general view in the present study, however, is that the optimality principle governing the relation between f and r in the cardiovascular system in general is not necessarily the same as nor necessarily related to the optimality principle governing branching angles. It is therefore believed to be unwise to make assumptions based on both of these principles, at this stage, before they have each been examined separately in the light of experimental evidence. Experimental data on this aspect of the cardiovascular system, there-
fore, can make a considerable contribution to our understanding of the design and function of this system.

Finally, we must emphasize the limitations of the mathematical model used in this study, particularly the assumptions on which Eq. 5 rests. The validity of these assumptions depends on the manner in which blood vessels are actually tethered in the neighborhood of branch points, a question which calls for much experimental study. Also, the neglect of such factors as the pulsatile nature of blood flow and the elasticity, nonuniformity, and longitudinal curvature of the vessel walls is clearly a measure of the simplicity and lack of sophistication of our model. However, apart from the immense mathematical difficulties which they would entail, such measures of sophistication can hardly be justified at this stage when only the rudimentary foundations of the model are being laid down.

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